Continuous Linear Transformation

Let N and N' be NLS. A linear transformation $T : N \to N'$ is continuous iff for any seq^n $\langle x_n \rangle$ in N converging to $x \in N$, the seq^n $\langle T(x_n) \rangle$ in N' converges to $T(x)$ in N'.

Bounded Linear Transformation

Let $T : N \to N'$ be a L.T.

N & $N' \to$ NLS

If $\exists$ a real number $K \geq 0$ s.t.

$\|T(x)\| \leq K \|x\| \quad \forall x \in N$

then K is called a bound for T and T is called a bounded linear transformation.

Thm 1. Let T be a L.T. of a ~~normed~~ normed linear space N into another normed linear space N'. Then the following statements are equivalent —

P.249

(i) T is continuous

(ii) T is continuous at the origin i.e.

$x_n \to 0 \Rightarrow T(x_n) \to 0$

(iii) $\exists$ a real number $K \geq 0$ s.t.

$$\|T(x)\| \leq K\|x\| \quad \forall x \in N$$

i.e. T is bounded

(iv) If $S = \{x \mid \|x\| \leq 1\}$ is the closed unit sphere in N, then its image is a bounded set in N'.

Proof: (i) $\Leftrightarrow$ (ii) Let T be continuous and $\langle x_n \rangle$ be any seq^n in N s.t.

$$x_n \to 0 \text{ as } n \to \infty$$

$$\therefore T(x_n) \to T(0) = 0$$

$\Rightarrow$ T is continuous at origin.

Converse. Let T be cont. at origin.

To show that T is continuous on N.

Let $\langle x_n \rangle$ be a seq^n in N s.t. $x_n \to x$; $x \in N$

Then $\to$

$$(x_n - x) \to 0$$

$\Rightarrow$ $T(x_n - x) = T(0)$, $\because$ T is cont. at origin

$\Rightarrow$ $T(x_n) - T(x) = 0$ $\because$ T is linear

$\Rightarrow$ $T(x_n) = T(x)$

$\Rightarrow$ T is cont. on N.

(ii) $\Leftrightarrow$ (iii) Let T be cont. at origin.

To show that T is bounded.

We shall prove it by contradiction.

Suppose T is not bounded i.e. $\nexists$ no real number K s.t.

$\|T(x)\| \leq K\|x\| \quad \forall x \in N$

Then we can find some x_n, for some $n \in I^+$ s.t.

$$\|T(x_n)\| > n\|x_n\|$$

$$\Rightarrow \frac{1}{n\|x_n\|}\|T(x_n)\| > 1$$

$$\Rightarrow \left\|\frac{1}{n\|x_n\|} T(x_n)\right\| > 1$$

$$\Rightarrow \left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| > 1 \longrightarrow (1)$$

$\because \frac{1}{n\|x_n\|}$ is a scalar

& $T(\alpha x) = \alpha T(x)$

Now let $\frac{x_n}{n\|x_n\|} = y_n$

Then $\|y_n\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} \to 0$ as $n \to \infty$

$\therefore y_n \to 0$ as $n \to \infty$

But $T(y_n) \not\to 0$ as from (1) $\to$

$\|T(y_n)\| > 1$

$\therefore T$ is not cont. at origin.

which is a contradiction.

Hence T must be bounded

<u>Converse</u> Let T be bounded. To show that T is cont. at origin.

Since T is bdd, $\therefore \exists$ a real number $K > 0$ s.t. $\|T(x)\| \leq K\|x\| \quad \forall x \in N$

$\longrightarrow (2)$

Now, let $<x_n>$ be a seq^n in N such that (8)
$x_n \to 0$. Then
$$\|x_n\| \to \|0\| = 0 \quad \to (3)$$
From (2),
$$\|T(x_n)\| \leq K\|x_n\| \quad \forall n \quad \to (4)$$
From (3) & (4) $\Rightarrow$
$$\|T(x_n)\| \leq 0$$
But $\|\cdot\|$ is always non-negative
$\therefore$ we must have
$$\|T(x_n)\| \to 0$$
i.e. $x_n \to 0 \Rightarrow T(x_n) \to 0$

$\therefore$ T is cont. at origin.

(iii) $\Leftrightarrow$ (iv) Let T be bdd.
Then $\|T(x)\| \leq K\|x\| \quad \forall x \in N \to (5)$
Let $x \in S$ where $S = \{x \mid \|x\| \leq 1\}$
Then $\|x\| \leq 1$
$\therefore$ $\|T(x)\| \leq K$, from (5)
$\forall x \in S$

$\therefore$ $T(S)$ is bdd i.e. image of S is bdd in N'.

<u>Converse</u> Let $T(S)$ be bdd. To show that T is bdd.
Since $T(S)$ is bdd., $\therefore$ we have
$$\|T(x)\| \leq K \quad \forall x \in S, \text{ for some real number } K > 0$$
$\to (6)$

Now we shall show that

$$\|T(x)\| \leq K\|x\|, \quad \forall x \in N$$

If $x=0$ then $Tx=0 \Rightarrow \|T(x)\| \leq K\|x\|$

If $x \neq 0$ then $\frac{x}{\|x\|} \in S$ (why?)

Coz $\left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$

$\therefore$ From eqn (6), we have

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq K$$

$$\Rightarrow \left\|\frac{1}{\|x\|} T(x)\right\| \leq K, \quad \because T(\alpha x) = \alpha T(x)$$

$$\Rightarrow \frac{1}{\|x\|}\|T(x)\| \leq K$$

$$\Rightarrow \|T(x)\| \leq K\|x\|, \quad \forall x \in N$$

$\therefore$ T is bounded.

(Proved)

Note: From above theorem, we can say that if a linear transformation is continuous then it is bounded and vice-versa

Now we shall define norm for such continuous (bounded) linear transformations.